

FUSION RULES ON A PARAMETRIZED SERIES OF GRAPHS

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ABSTRACT. A series of pairs of graphs (Γ_k, Γ'_k) , $k = 0, 1, 2, \dots$ has been considered as candidates for dual pairs of principal graphs of subfactors of small Jones index above 4 and it has recently been proved that the pair (Γ_k, Γ'_k) comes from a subfactor if and only if $k = 0$ or $k = 1$. We show that nevertheless there exists a unique fusion system compatible with this pair of graphs for all non-negative integers k .

1. INTRODUCTION

A subfactor $N \subset M$ with finite index and finite depth generates finitely many isomorphism classes of bimodules with four different combinations of left and right coefficients. They form a bi-graded fusion category. Its Grothendieck ring form a *fusion ring* or a *fusion hypergroup*, namely a bi-graded \mathbb{Z} -algebra \mathcal{A} with following properties:

- it has a basis given by finitely many irreducible bimodules of four different kinds $\mathcal{X} = {}_N\mathcal{X}_N \sqcup {}_N\mathcal{X}_M \sqcup {}_M\mathcal{X}_N \sqcup {}_M\mathcal{X}_M$ (we call the labels N, M right or left coefficients, depending on the position),
- an involution $X \in {}_P\mathcal{X}_Q \rightarrow \overline{X} \in {}_Q\mathcal{X}_P$ is defined, where $P, Q \in \{N, M\}$.
- a product is defined for a pair of bimodules with “matching” coefficient, namely, for a pair $(X, Y) \in \mathcal{X} \times \mathcal{X}$ such that the right coefficient of X and the left coefficient of Y match, XY is defined. It decomposes as follows:

$$XY = \sum N_{X,Y}^Z Z,$$

where the sum is taken over those $Z \in \mathcal{X}$ that have the same left (resp. right) coefficient as X (resp. Y), and $N_{X,Y}^Z \in \mathbb{N}_0$, moreover Frobenius reciprocity holds:

$$N_{X,Y}^Z = N_{Z,\overline{Y}}^X = N_{\overline{X},Z}^Y = N_{\overline{Y},\overline{X}}^{\overline{Z}} = N_{\overline{Z},X}^{\overline{Y}} = N_{Y,\overline{Z}}^{\overline{X}}.$$

- There are identity objects $\mathbf{1}_N \in {}_N\mathcal{X}_N$, $\mathbf{1}_M \in {}_M\mathcal{X}_M$ that act as identity with respect to the product, whenever it is defined.

The involution extends linearly to define an involution on \mathcal{A} . For a fusion ring \mathcal{A} , there is a unique weight function $\mu : \mathcal{A} \rightarrow \mathbb{R}_{\geq}$ satisfying

$$\begin{aligned} \mu(\mathbf{1}_N) &= \mu(\mathbf{1}_M) = 1, \\ \mu(XY) &= \mu(X)\mu(Y), \end{aligned}$$

where $X, Y, Z \in \mathcal{X}$ are with suitable coefficients for each equality, so that XY and $X + Z$ are defined. The *(dual) principal graph* of the subfactor encode partial information of the fusion

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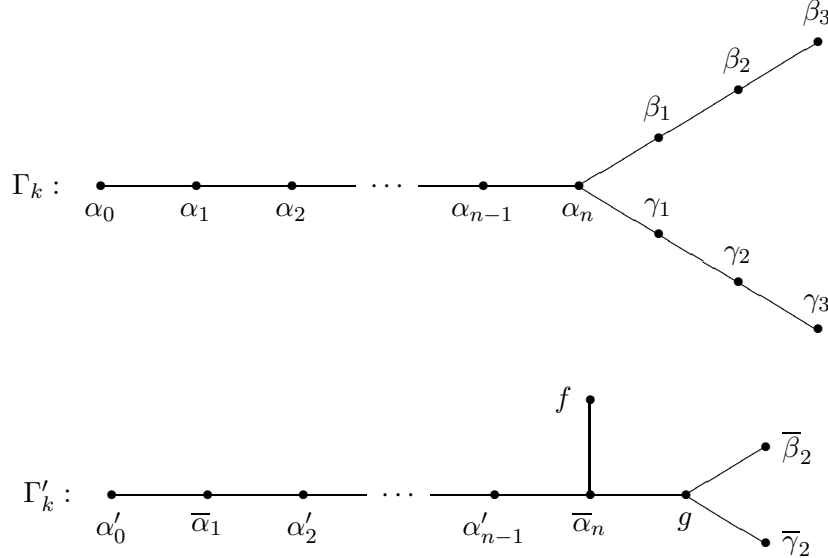
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algebra: namely, the (dual) principal graph has the vertices corresponding to ${}_N\mathcal{X}_N \sqcup {}_N\mathcal{X}_M$ (resp. ${}_M\mathcal{X}_N \sqcup {}_M\mathcal{X}_M$), with the number of the edges between vertices ${}_NX_N$ and ${}_NY_M$ (resp. ${}_MX_M$ and ${}_MY_N$) given by N_{X,NM_M}^Y (resp. N_{X,MN_N}^Y .)

On the other hand, one may start with a pair of graphs, and may consider if there is a fusion algebra compatible with the fusion constraints determined by the graphs. Such investigation may be used to exclude graphs as (dual) principal graphs of subfactors. For example, type E_7 and D_{2n+1} Dynkin diagrams are proved *not* to be (dual) principal graphs of subfactors, by showing that the fusion constraints given by the graphs give rise to inconsistency in fusion rules ([8], [9]). Note that the existence of a fusion algebra compatible with a given pair of graphs do not imply the existence of a subfactor with given graphs as (dual) principal graphs.

In this paper, we deal with the following series of pairs of graphs:



where $n = 4k + 3$, $k = 0, 1, \dots$. Let These graphs are a part of the list of the graphs that were candidates for (dual) principal graphs of a subfactor with indices between 4 and $3 + \sqrt{3}$ given by the second author ([6]). Note that the notation used here is somewhat different from the one used in [6]. It has been already proved that, for $k = 0, 1$, Γ_k (resp. Γ'_k) are (dual) principal graphs of a subfactors ([2], [4]), and for $k > 1$, they are not realized as (dual) principal graphs ([3]). In this paper, we prove that, despite that Γ_k (resp. Γ'_k) are not principal graphs for $k > 1$, there are still fusion algebras consistent with the graphs, and moreover such fusion algebras are unique for each k . Namely we prove the following:

Theorem 1.1. *Let $V_{11} := \{\text{even vertices of } \Gamma_k\}$, $V_{12} := \{\text{odd vertices of } \Gamma_k\}$, $V_{21} := \{\text{odd vertices of } \Gamma'_k\}$, $V_{22} := \{\text{even vertices of } \Gamma'_k\}$, and $V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$. For each k , there is a unique fusion algebra $\mathcal{A} = \mathbb{Z}\mathcal{X}$, where*

$$\mathcal{X} = {}_N\mathcal{X}_N \sqcup {}_N\mathcal{X}_M \sqcup {}_M\mathcal{X}_N \sqcup {}_M\mathcal{X}_M$$

compatible with the graphs Γ_k, Γ'_k . Namely

$${}_N\mathcal{X}_N = V_{11},$$

$${}_N\mathcal{X}_M = V_{12},$$

$${}_M\mathcal{X}_N = V_{21},$$

$${}_M\mathcal{X}_M = V_{22}$$

as sets, and

$$N_{X,\alpha_1}^Y (\text{resp. } N_{X,\bar{\alpha}_1}^Y) = \begin{cases} 1 & \text{if } X \text{ and } Y \text{ are connected by an edge} \\ 0 & \text{else,} \end{cases}$$

$$N_{X,1}^Y = \delta_{X,Y}$$

holds, where $X, Y \in \mathcal{X}$, and 1 denotes identity objects $1_N = \alpha_0 \in {}_N\mathcal{X}_N$ or $1_M = \alpha'_0 \in {}_M\mathcal{X}_M$.

The content of this paper is as follows. In Section 2 we show that if there is a fusion system compatible with the graphs Γ_k, Γ'_k , it must be unique. In Section 3 we show the existence of such a fusion system.

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2. UNIQUENESS, POSITIVITY, AND INTEGRALITY OF THE FUSION RULES

In this section we prove that, if there is a fusion algebra compatible with the graphs, it is unique. Positivity and integrality of fusion coefficients is derived: we do not impose them in showing uniqueness of the fusion rules.

2.1. Fusion rules for the even vertices. In this subsection we show that there is a unique fusion algebra structure on $\mathcal{A}_1 = \mathbb{Z}_N\mathcal{X}_N$ compatible with the graph Γ_k . The main issue is to determine the fusion rule among $\beta_1, \beta_3, \gamma_1, \gamma_3$. The rest will follow easily from this.

In the following we assume that there is a fusion algebra compatible with (Γ_k, Γ'_k) . The involution $\gamma \in V \rightarrow \bar{\gamma} \in V$ extends linear to a map on $\mathbb{R}V$. For simplicity, we refer to the objects in \mathcal{X} by corresponding vertices in V . For $X := \sum N_X^Z Z \in \mathbb{R}V$ and $Y \in V$, we denote

$$\langle X, Y \rangle = \langle Y, X \rangle := N_X^Y.$$

Observe that $\langle \cdot, \cdot \rangle$ extends linearly to define a bilinear form on $\mathbb{R}V$, and

$$\langle XY, Z \rangle = \langle X, Z\bar{Y} \rangle = \langle Y, \bar{X}Z \rangle$$

holds by Frobenius reciprocity. The graph Γ_k encodes the decomposition of $X\alpha_1$ for X in V_{11} into a direct sum of vertices from V_{12} and the decomposition of $Y\bar{\alpha}_1$ into a direct sum of vertices from V_{11} . Let G be the adjacency matrix for (V_{11}, V_{12}) , namely

$$G = (G_{X,Y})_{X \in V_{11}, Y \in V_{12}},$$

where $G_{X,Y}$ = (the number of the edges connecting X and Y)

$$= \langle X\alpha_1, Y \rangle$$

$$= \langle Y\bar{\alpha}_1, X \rangle,$$

which is written as the following $(\frac{n+1}{2} + 4) \times (\frac{n+1}{2} + 2)$ -matrix :

$$G = \begin{matrix} & \beta_2 & \gamma_2 & \alpha_n & \alpha_{n-2} & \cdots & \cdots & \alpha_1 \\ \begin{matrix} \beta_3 \\ \beta_1 \\ \gamma_3 \\ \gamma_1 \\ \alpha_{n-1} \\ \vdots \\ \alpha_2 \\ \alpha_0 \end{matrix} & \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 1 \end{array} \right) \end{matrix}, \quad (1)$$

Let

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},$$

then

$$\Delta^2 = \begin{pmatrix} GG^t & 0 \\ 0 & G^tG \end{pmatrix}.$$

We put $\mathbb{D} := GG^t$, which acts on $\overline{\mathcal{A}}_1 := \mathbb{R}V_{11}$. We utilize certain eigen vectors of \mathbb{D} to determine the fusion structure of \mathcal{A}_1 .

Observe from the graph that

$$\begin{aligned} \Delta\beta_1 &= \alpha_n + \beta_2, & \Delta\gamma_1 &= \alpha_n + \gamma_2, \\ \Delta\beta_2 &= \beta_1 + \beta_3, & \Delta\gamma_2 &= \gamma_1 + \gamma_2, \\ \Delta\beta_3 &= \beta_2, & \Delta\gamma_3 &= \gamma_2. \end{aligned}$$

Put

$$\begin{aligned} \xi &= (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3) \\ \eta &= (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3). \end{aligned}$$

Then

$$\begin{aligned} \mathbb{D}\xi &= \Delta^2\xi = \Delta(2\beta_2 - 2\gamma_2) = 2\xi, \\ \mathbb{D}\eta &= \Delta^2\eta = 0. \end{aligned}$$

Let $E(\mathbb{D}, c)$, $c \in \mathbb{R}$ be the eigenspace of the eigenvalue c for \mathbb{D} in $\mathbb{R}(V_{11})$.

Lemma 2.1.

$$\dim E(\mathbb{D}, 2) = E(\mathbb{D}, 0) = 2$$

Proof

$$\mathbb{D} = \begin{matrix} & \begin{matrix} \beta_3 & \beta_1 & \gamma_3 & \gamma_1 & \alpha_{n-1} & \cdots & \cdots & \cdots & \alpha_2 & \alpha_0 \end{matrix} \\ \begin{matrix} \beta_3 \\ \beta_1 \\ \gamma_3 \\ \gamma_1 \\ \alpha_{n-1} \\ \alpha_{n-3} \\ \vdots \\ \vdots \\ \alpha_2 \\ \alpha_0 \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 2 & 0 & 1 & 1 & 0 & & & & \vdots \\ 0 & 0 & 1 & 1 & 0 & 0 & & & & \vdots \\ 0 & 1 & 1 & 2 & 1 & 0 & & & & \vdots \\ 0 & 1 & 0 & 1 & 2 & 1 & 0 & & & \vdots \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & 0 & 1 & 2 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{pmatrix} \end{matrix}$$

Recall that $n = 4k + 3$. Let $\rho_k(x) := \det(tI - \mathbb{D})$ be the characteristic polynomial of $\mathbb{D} = GG^t$. It was proved in [1] that the characteristic polynomial of G^tG is equal to $(t - 2)^2 q_k(t)$, where the polynomials $q_k(t), k \geq 0$, can be defined recursively by

$$\begin{aligned} q_0(t) &= t^2 - 5t + 3 \\ q_1(t) &= (t - 1)(t^3 - 8t^2 + 17t - 5) \\ q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2 \end{aligned}$$

Since the matrix G has $2k + 6$ rows and $2k + 4$ columns, GG^t is a unitary conjugate of $G^tG \oplus O_2$, where O_2 is the zero 2×2 matrix. Hence

$$\begin{aligned} \rho_k(t) &= t^2 \det(tI - G^tG) \\ &= t^2(t - 2)^2 q_k(t). \end{aligned}$$

Using the recursion formula for $q_k(t)$, one gets $q_k(0) = 2k + 3$ and $q_k(2) = (-1)^k 2(k + 1)(2k + 3)$. In particular neither 0 nor 2 is a root of q_k . Hence 0 and 2 are roots of multiplicity 2 in ρ_k . Since $\mathbb{D} = GG^t$ is a symmetric matrix, it follows that the dimensions of the eigenspaces for \mathbb{D} for the eigenvalues 0 and 2 are both equal to two.

Bases of $E(\mathbb{D}, 2)$, $E(\mathbb{D}, 0)$ may be taken as follows:

$$\begin{aligned} E(\mathbb{D}, 2) &:= \text{span}\{x_1, x_2\} \\ E(\mathbb{D}, 0) &:= \text{span}\{y_1, y_2\}, \end{aligned}$$

where

$$\begin{aligned} x_1 &:= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2}) \\ &\quad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3) \\ x_2 &:= \xi = (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3) \\ y_1 &:= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3) \\ y_2 &:= \eta = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3) \end{aligned}$$

Assume that we have a fusion algebra compatible with the pair of the graphs (Γ_k, Γ'_k) , and let π and π' be the conjugate maps $\gamma \mapsto \bar{\gamma}$ on V_{11} and V_{22} . By the argument used in [6, pp28-31], π' fixes every element of V_{22} . For π , there are only two possibilities:

Case 1 (=case (b) in [6, p31])

$$\bar{\beta}_1 = \beta_1, \bar{\gamma}_1 = \gamma_1, \bar{\beta}_3 = \gamma_3 (\Leftrightarrow \bar{\gamma}_3 = \beta_3),$$

Case 2 (=case (a) in [6, p31]. To be eliminated.)

$$\bar{\beta}_1 = \gamma_1 (\Leftrightarrow \bar{\gamma}_1 = \beta_1), \bar{\beta}_3 = \beta_3, \bar{\gamma}_3 = \gamma_3.$$

In both cases, $\bar{\alpha}_{2j} = \alpha_{2j}$ for $j = 0, 1, \dots, 2k+1$. Note that π extends linearly to \mathcal{A}_1 and $\bar{\mathcal{A}}_1 = \mathbb{R}V_{11}$. Let $E(\mathbb{D}, c)_{sc} := E(\mathbb{D}, c)^\pi$. Observe that

$$c_1 \bar{x}_1 + c_2 \bar{x}_2 = c_1 x_1 + c_2 x_2, \quad c_1, c_2 \in \mathbb{R}$$

holds if and only if $c_2 = 0$ in both cases 1 and 2, and similarly

$$c_1 c_1 \bar{y}_1 + c_2 \bar{y}_2 = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in \mathbb{R}$$

if and only if $c_2 = 0$ in both cases. Therefore

$$\begin{aligned} E(\mathbb{D}, 2)_{sc} &= \mathbb{R}x_1 \\ E(\mathbb{D}, 0)_{sc} &= \mathbb{R}y_1 \end{aligned}$$

By the definition of principal graphs, the matrix $\mathbb{D} : \mathbb{R}V_{11} \rightarrow \mathbb{R}V_{11}$ corresponds to the fusion rule of the right tensor product by $\alpha\bar{\alpha}$, where $\alpha = \alpha_1$. Therefore

$$\begin{aligned} \mathbb{D}(\bar{\xi}\xi) &= \bar{\xi}\mathbb{D}(\xi) = 2\bar{\xi}\xi \\ \mathbb{D}(\bar{\eta}\eta) &= \bar{\eta}\mathbb{D}(\eta) = 0. \end{aligned}$$

Hence

$$\begin{aligned} \bar{\xi}\xi &\in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1, \\ \bar{\eta}\eta &\in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1. \end{aligned}$$

Thus

$$\langle \bar{\xi}\xi, \alpha_0 \rangle = \langle \xi, \xi \alpha_0 \rangle = \langle \xi, \xi \rangle = 4.$$

Hence the coefficient of $\bar{\xi}\xi$ at α_0 is 4. Since $\bar{\xi}\xi \in \mathbb{R}x_1$, we have $\bar{\xi}\xi = 2x_1$. Likewise we obtain $\bar{\eta}\eta = 2y_1$. Noting that

$$\bar{\xi} = \begin{cases} \eta & \text{for Case 1} \\ -\eta & \text{for Case 2,} \end{cases}$$

we have

- In Case 1: $\xi\eta = 2y_1, \eta\xi = 2x_1$,
- In Case 2: $\xi\eta = -2y_1, \eta\xi = -2x_1$.

Lemma 2.2.

$$\xi^2 = 0, \eta^2 = 0.$$

Proof. Since $\mathbb{D}(\xi^2) = \xi\mathbb{D}(\xi) = 2\xi^2$, $\xi^2 = c_1x_1 + c_2x_2$ for some $c_1, c_2 \in \mathbb{R}$. Moreover, since $\langle \xi, \eta \rangle = 0$, we have

$$\begin{aligned} \langle \xi^2, \alpha_0 \rangle &= \langle \xi, \bar{\xi}\alpha_0 \rangle = \pm \langle \xi, \eta \rangle \\ &= 0 \end{aligned}$$

Together with $\langle c_1x_1 + c_2x_2, \alpha_0 \rangle = 2c_1$, $c_1, c_2 \in \mathbb{R}$, we obtain

$$\xi^2 = c_2x_2 = c_2\xi.$$

We show that $c_2 = 0$:

$$\begin{aligned} 4c_2 &= \langle c_2\xi, c_2\xi \rangle = \langle \xi^2, \xi^2 \rangle = \langle \bar{\xi}\xi, \xi\bar{\xi} \rangle = 4 \langle x_1, y_1 \rangle \\ &= (2 - 2) - (2 - 2) + \cdots (-1)^k(2 - 2) + (1 + 1 - 1 - 1) = 0. \end{aligned}$$

We used that $\bar{\xi}\xi = 2x_1$, $\xi\bar{\xi} = 2y_1$ for both cases. Thus $\xi^2 = 0$. Then $\bar{\xi}^2 = \eta^2 = 0$ for both cases. \square

Since $\beta_3 - \gamma_3 = \frac{1}{2}(\xi - \eta)$, we get

$$\begin{aligned} (\beta_3 - \gamma_3)^2 &= \frac{1}{4}(\xi - \eta)^2 \\ &= \frac{1}{4}(\xi^2 + \eta^2 - \xi\eta - \eta\xi) \\ &= -\frac{1}{4}(\xi\eta + \eta\xi) \\ &= \begin{cases} -\frac{1}{2}(x_1 + y_1) & \text{in Case 1} \\ \frac{1}{2}(x_1 + y_1) & \text{in Case 2} \end{cases} \end{aligned}$$

Remark 2.3. For k even (i.e. $n = 3 \pmod{8}$) and $k = 2l$,

$$\frac{1}{2}(x_1 + y_1) = 2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l}) - (\beta_3 + \gamma_3)$$

and for k odd (i.e. $n = 7 \pmod{8}$) and $k = 2l + 1$,

$$\begin{aligned} \frac{1}{2}(x_1 + y_1) &= 2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1 + \gamma_1) \end{aligned}$$

Consider next the sequence of polynomials R_n given recursively by

$$\begin{aligned} R_0(t) &= 1, \\ R_1(t) &= t, \\ R_m(t) &= tR_{m-1}(t) - R_{m-2}(t), n \geq 2. \end{aligned}$$

as in [6, p33–34]. Note that $R_m(t) = U_m(\frac{t}{2})$, where U_m is the m -th Chebyshev polynomial of second kind [5]. Moreover,

$$R_m(2 \cos \theta) = \frac{\sin(m+1)\theta}{\sin \theta}, \quad 0 < \theta < \pi.$$

By the recursion formula for R_n , it follows that

$$\begin{aligned} R_j(\Delta)\alpha_0 &= \alpha_j, \quad 0 \leq j \leq n, \\ R_{n+1}(\Delta)\alpha_0 &= \beta_1 + \gamma_1, \\ R_{n+2}(\Delta)\alpha_0 &= \alpha_n + \beta_2 + \gamma_2, \\ R_{n+3}(\Delta)\alpha_0 &= \alpha_{n-1} + \beta_1 + \gamma_1 + \beta_3 + \gamma_3. \end{aligned}$$

Hence

$$\begin{aligned} \beta_3 + \gamma_3 &= (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))\alpha_0 \\ &= (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))\alpha_0 \end{aligned}$$

For m even, $R_m(t)$ is an even polynomial in t , thus there is are unique polynomials $(Q_j)_{j=0,1,2,\dots}$ with $\deg(Q_l) = l$, such that

$$Q_j(t^2) = R_{2j}(t), \quad t \in \mathbb{R}, \quad j = 0, 1, 2, \dots$$

With this notation, we have

$$\begin{aligned} \beta_3 + \gamma_3 &= (Q_{2k+3}(\mathbb{D}) - Q_{2k+2}(\mathbb{D}) - Q_{2k+1}(\mathbb{D}))\alpha_0 \\ &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha\overline{\alpha}). \end{aligned}$$

Therefore

$$\begin{aligned} (\beta_3 - \gamma_3)(\beta_3 + \gamma_3) &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\mathbb{D})(\beta_3 - \gamma_3) \\ &= \frac{1}{2}(Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\mathbb{D})(\xi - \eta) \end{aligned}$$

Since $\mathbb{D}\xi = 2\xi$ and

$$\begin{aligned} Q_m(2) = R_{2j}(\sqrt{2}) &= \frac{\sin(2j+1)\pi/4}{\sin \pi/4} \\ &= \begin{cases} 1, & j = 0, 1 \pmod{4} \\ -1, & j = 2, 3 \pmod{4}, \end{cases} \end{aligned}$$

we have

$$Q_j(\mathbb{D})\xi = \begin{cases} \xi, & j = 0, 1 \pmod{4} \\ -\xi, & j = 2, 3 \pmod{4}, \end{cases}$$

Similarly, since $\mathbb{D}\eta = 0$ and

$$Q_j(0) = R_{2j}(0) = \frac{\sin(2j+1)\pi/2}{\sin \pi/2} = (-1)^j,$$

we have

$$Q_j(\mathbb{D})\eta = (-1)^j\eta, \quad j = 0, 1, 2, \dots$$

Therefore we have

$$\begin{aligned} &(Q_{2k+3}(\mathbb{D}) - Q_{2k+2}(\mathbb{D}) - Q_{2k+1}(\mathbb{D}))\xi \\ &= \begin{cases} (Q_{4l+3}(\mathbb{D}) - Q_{4l+2}(\mathbb{D}) - Q_{4l+1}(\mathbb{D}))\xi = -\xi & \text{for } k = 2l, l \in \mathbb{N}_0 \\ (Q_{4l+5}(\mathbb{D}) - Q_{4l+4}(\mathbb{D}) - Q_{4l+3}(\mathbb{D}))\xi = \xi & \text{for } k = 2l + 1, l \in \mathbb{N}_0, \end{cases} \end{aligned}$$

and in both cases

$$(Q_{2k+3}(\mathbb{D}) - Q_{2k+2}(\mathbb{D}) - Q_{2k+1}(\mathbb{D}))\eta = -\eta.$$

Hence

$$\begin{aligned} (\beta_3 - \gamma_3)(\beta_3 + \gamma_3) &= \frac{1}{2}(Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\mathbb{D})(\xi - \eta) \\ &= \begin{cases} \frac{1}{2}(-\xi + \eta) = \gamma_3 - \beta_3, & k \text{ even}, \\ \frac{1}{2}(\xi + \eta) = \beta_1 - \gamma_1, & k \text{ odd}. \end{cases} \end{aligned}$$

Using the contragradient map we get

For Case 1:

$$\begin{aligned} (\beta_3 + \gamma_3)(\beta_3 - \gamma_3) &= \overline{(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)} \\ &= \overline{(\gamma_3 - \beta_3)(\gamma_3 + \beta_3)} \\ &= -\overline{(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)} \\ &= \begin{cases} -(\bar{\gamma}_3 - \bar{\beta}_3) = -(\beta_3 - \gamma_3), & k \text{ even}, \\ -(\bar{\beta}_1 - \bar{\gamma}_1) = -(\beta_1 - \gamma_1), & k \text{ odd}, \end{cases} \end{aligned}$$

For Case 2 (to be eliminated):

$$\begin{aligned} (\beta_3 + \gamma_3)(\beta_3 - \gamma_3) &= \overline{(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)} \\ &= \overline{(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)} \\ &\begin{cases} \bar{\gamma}_3 - \bar{\beta}_3 = \gamma_3 - \beta_3, & k \text{ even}, \\ \bar{\beta}_1 - \bar{\gamma}_1 = \gamma_1 - \beta_1, & k \text{ odd}. \end{cases} \end{aligned}$$

Thus in both cases

$$(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = \begin{cases} \gamma_3 - \beta_3, & k \text{ even}, \\ \gamma_1 - \beta_1, & k \text{ odd}. \end{cases}$$

So far, we have obtained the following three formulae:

[A]

$$(\beta_3 - \gamma_3)^2 = \begin{cases} -\frac{1}{2}(x_1 - y_1) & \text{in Case 1} \\ \frac{1}{2}(x_1 - y_1) & \text{in Case 2} \end{cases}$$

[B]

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = \begin{cases} \frac{1}{2}(-\xi + \eta) = \gamma_3 - \beta_3, & k \text{ even}, \\ \frac{1}{2}(\xi + \eta) = \beta_1 - \gamma_1, & k \text{ odd}. \end{cases}$$

[C]

$$(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = \begin{cases} \gamma_3 - \beta_3, & k \text{ even}, \\ \gamma_1 - \beta_1, & k \text{ odd}. \end{cases}$$

Next we compute $(\beta_3 + \gamma_3)^2$, in order to find β_3^2 , γ_3^2 , $\beta_3\gamma_3$ and $\gamma_3\beta_3$.

Claim 2.4.

[D] $(\beta_3 + \gamma_3)^2 = 2(c_0\alpha_0 + c_1\alpha_2 + \cdots + c_{2k+1}\alpha_{4k+2}) + c_{2k+2}(\beta_1 + \gamma_1) + c_{2k}(\beta_3 + \gamma_3)$,
 where c_j 's are defined by $c_0 = 1$, $c_1 = c_2 = 0$ and $c_j = c_{j-1} + c_{j-2} + c_{j-3}$ for $j \geq 3$.

Proof

Recall that

$$\begin{aligned} (\beta_3 + \gamma_3) &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\mathbb{D})\alpha_0 \\ &= (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))\alpha_0, \end{aligned}$$

thus

$$(\beta_3 + \gamma_3)^2 = (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))(\beta_3 + \gamma_3). \quad \cdots (\sharp)$$

Our strategy of the proof is as follows: first we find a sequence of polynomials (S_j) such that $S_j(\Delta)(\beta_3 + \gamma_3)$ is given by a simple formula. Next we rewrite the right hand side of (\sharp) using (S_j) 's.

Observe that we obtain from the graph the following:

$$\begin{aligned} R_0(\Delta)(\beta_3 + \gamma_3) &= (\beta_3 + \gamma_3), \\ R_1(\Delta)(\beta_3 + \gamma_3) &= (\beta_2 + \gamma_2), \\ R_2(\Delta)(\beta_3 + \gamma_3) &= \Delta(\beta_2 + \gamma_2) - (\beta_3 + \gamma_3) = \beta_1 + \gamma_1, \\ R_3(\Delta)(\beta_3 + \gamma_3) &= \Delta(\beta_1 + \gamma_1) - (\beta_2 + \gamma_2) = 2\alpha_n, \\ R_4(\Delta)(\beta_3 + \gamma_3) &= 2\Delta\alpha_n - (\beta_1 + \gamma_1) = 2\alpha_{n-1} + \beta_1 + \gamma_1, \end{aligned}$$

We define the polynomials $(S_j(t))_{j \geq 3}$ by the following recursive formula:

$$\begin{aligned} S_3(t) &= R_3(t), \\ S_4(t) &= R_4(t) - R_2(t), \\ S_j(t) &= tS_{j-1}(t) - S_{j-2}(t), \quad j \geq 5. \end{aligned}$$

By definition $S_3(\Delta)(\beta_3 + \gamma_3) = 2\alpha_n$, $S_4(\Delta)(\beta_3 + \gamma_3) = 2\alpha_{n-1}$. Since $\alpha_{l-1} = \Delta\alpha_l - \alpha_{l+1}$ for $l = 1, 2, \dots, n-1$, we easily obtain

$$S_j(\Delta)(\beta_3 + \gamma_3) = 2\alpha_{n-j+3}$$

for $j = 3, 4, \dots, n+3$. Next we express R_j 's in terms of S_j 's.

Lemma 2.5. For $j \geq 2$,

$$\begin{aligned} R_{2j-1} &= d_0S_{2j-1} + d_1S_{2j-3} + \cdots + d_{j-2}S_3 + (d_{j-1} - d_{j-2})R_1 \\ R_{2j} &= d_0S_{2j} + d_1S_{2j-2} + \cdots + d_{j-2}S_4 + d_{j-1}R_2 + d_{j-3}R_0, \end{aligned}$$

where d_j 's satisfy

$$\begin{aligned} d_j &= d_{j-1} + d_{j-2} + d_{j-3} \\ d_{-1} &= 0, d_0 = d_1 = 1, \end{aligned}$$

Proof of Lemma:

For $j = 2$ it is obvious by the definition of S_j 's. We proceed with induction. Assume that it is

true for j ($j \geq 2$). Using the recursion formulae for R_j 's and S_j 's, we have

$$\begin{aligned}
R_{2j+1}(t) &= tR_{2j}(t) - R_{2j-1}(t) \\
&= t(d_0S_{2j} + d_1S_{2j-2} + \cdots + d_{j-2}S_4 + d_{j-1}R_2 + d_{j-3}) \\
&\quad - (d_0S_{2j-1} + d_1S_{2j-3} + \cdots + d_{j-2}S_3 + (d_{j-1} - d_{j-2})R_1) \\
&= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + t(d_{j-1}R_2 + d_{j-3}) - (d_{j-1} - d_{j-2})R_1 \\
&= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + d_{j-1}(tR_2 - R_1) + td_{j-3} - d_{j-2}R_1 \\
&= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + d_{j-1}S_3 + (d_{j-3} - d_{j-2})R_1.
\end{aligned}$$

The last equality was obtained using $S_3 = R_3$, $R_1 = t$, and $d_{j-2} + d_{j-3} = d_j - d_{j-1}$. Likewise we have

$$\begin{aligned}
R_{2j+2}(t) &= tR_{2j+1}(t) - R_{2j}(t) \\
&= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 \\
&\quad + t(d_{j-1}S_3 + (d_j - d_{j-1})R_1) - (d_{j-1}R_2 + d_{j-3}R_0) \\
&= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}R_4 \\
&\quad + (d_j - d_{j-1})(R_2 + R_0) - d_{j-3}R_0 \\
&= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 + \\
&\quad + d_jR_2 + (d_j - d_{j-1} - d_{j-3})R_0 \\
&= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 + d_jR_2 + d_{j-2}R_0.
\end{aligned}$$

□

Let us go back to (\sharp) . Using Lemma 2.5,

$$\begin{aligned}
&R_{4k+6} - R_{4k+4} - R_{4k+2} \\
&= d_0S_{4k+6} + (d_1 - d_0)S_{4k+4} + d_{-1}S_{4k+2} + d_0S_{4k} + d_1S_{4k-2} + \cdots \\
&\quad + d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0 \\
&= S_{4k+6} + d_0S_{4k} + d_1S_{4k-2} + \cdots \\
&\quad + d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0
\end{aligned}$$

Recall

$$\begin{aligned}
S_j(\Delta)(\beta_3 + \gamma_3) &= 2\alpha_{n-j+3} \\
R_2(\beta_3 + \gamma_3) &= \beta_1 + \gamma_1.
\end{aligned}$$

Letting $c_0 := 1$, $c_1 = c_2 = 0$, $c_j := d_{j-3}$ for $j \geq 3$, we obtain the formula [D]. This concludes the proof for Claim 2.4. □

Thus far we obtained the formulae for $(\beta_3 - \gamma_3)^2$, $(\beta_3 - \gamma_3)(\beta_3 + \gamma_3)$, $(\beta_3 + \gamma_3)(\beta_3 - \gamma_3)$ and $(\beta_3 + \gamma_3)^2$ as in [A], [B], [C], [D]. This enable us to understand the fusion rules among β_3, γ_3 and their conjugates. We obtain the following:

Proposition 2.6. *The Case 2 does not occur. Namely β_1, γ_1 are self conjugate and $\overline{\beta}_3 = \gamma_3$ if there is a fusion algebra compatible with the graphs Γ_k, Γ'_k .*

Proof.

First observe that, by the definition $(c_j)_{j \geq 0}$ used in Claim 2.4, it follows that $c_j \pmod{4}$ is periodic in j with period 8. The values are given in the following Table 1:

TABLE 1.

$j \pmod{8}$	0	1	2	3	4	5	6	7
$c_j \pmod{4}$	1	0	0	1	1	2	0	0

In particular,

$$(\star) \begin{cases} c_{2j} = 1 \pmod{4} \text{ for } j \text{ even,} \\ c_{2j} = 0 \pmod{4} \text{ for } j \text{ odd.} \end{cases}$$

In the following we assume Case 2 and derive contradiction.

• **for k even:**

By [B] and [C], we have

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = (\beta_3 + \gamma_3)(\beta_3 - \gamma_3),$$

hence

$$\begin{aligned} \beta_3 \gamma_3 &= \gamma_3 \beta_3 = \frac{1}{2}(\beta_3 \gamma_3 + \gamma_3 \beta_3) \\ &= \frac{1}{4}((\beta_3 + \gamma_3)^2 - (\beta_3 - \gamma_3)^2). \end{aligned}$$

From [A] (Case 2), [D] and Remark 2.3, it follows that the coefficient of β_3 in the expansion of $\beta_3 \gamma_3$ in irreducible objects is equal to

$$\frac{c_{2k} + 1}{4}.$$

Since k is even, $c_{2k} = 1 \pmod{4}$ by (\star) , $(c_{2k} + 1)/4$ is not an integer. This implies that Case 2 does not occur if k is even.

• **for k odd:**

From [B], [C], we get

$$(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = -(\beta_3 + \gamma_3)(\beta_3 - \gamma_3).$$

Hence

$$\begin{aligned} \beta_3^2 &= \gamma_3^2 = \frac{1}{2}(\beta_3^2 + \gamma_3^2) \\ &= \frac{1}{4}((\beta_3 + \gamma_3)^2 + (\beta_3 - \gamma_3)^2). \end{aligned}$$

From [A] (Case 2), [D] and Remark 2.3, it follows that the coefficient of β_1 in the expansion of β_3^2 in irreducible objects is equal to

$$\frac{c_{2k+2} + 1}{4}.$$

Since k is odd, $c_{2k+2} = 1 \pmod{4}$ by (\star) , $(c_{2k+2} + 1)/4$ is not an integer. This excludes Case 2 for k odd as well. \square

In the following we determine all the irreducible decompositions for the products of any two objects in V , and show that the coefficients are non-negative integers. Since we excluded Case 2, we rewrite the formula [A]:

[A'] For $k = 2l$, $l = 0, 1, 2, \dots$,

$$(\beta_3 - \gamma_3)^2 = -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \dots + \alpha_{8l}) - (\beta_3 + \gamma_3),$$

and for $k = 2l + 1$, $l = 0, 1, 2, \dots$,

$$(\beta_3 - \gamma_3)^2 = -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \dots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1 + \gamma_1).$$

Put

$$\begin{aligned} A &:= (\beta_3 - \gamma_3)^2 \\ B &:= (\beta_3 - \gamma_3)(\beta_3 + \gamma_3) \\ C &:= (\beta_3 + \gamma_3)(\beta_3 - \gamma_3) \\ D &:= (\beta_3 + \gamma_3)^2. \end{aligned}$$

Then

$$\begin{aligned} \beta_3 \gamma_3 &= \frac{(D - A) + (B - C)}{4} \\ \gamma_3 \beta_3 &= \frac{(D - A) - (B - C)}{4} \\ \beta_3^2 &= \frac{(D + A) + (B + C)}{4} \\ \gamma_3^2 &= \frac{(D + A) - (B + C)}{4} \end{aligned}$$

We introduce new constants $(f_j)_{j \geq 0}$, $(g_j)_{j \geq 0}$ by

$$\begin{cases} f_j = \frac{1}{2}(c_j + 1), g_j = \frac{1}{2}(c_j - 1) & \text{when } j \equiv 0 \pmod{4}, \\ f_j = \frac{1}{2}(c_j - 1), g_j = \frac{1}{2}(c_j + 1) & \text{when } j \equiv 3 \pmod{4}, \\ f_j = g_j = \frac{1}{2}c_j & \text{when } j \equiv 1, 2 \pmod{4}. \end{cases}$$

Note that $f_j + g_j = c_j$ for all j . Furthermore, from Table 1, observe that f_j , g_j 's are non-negative integers for all $j \geq 0$. The list of some values for f_j 's and g_j 's are given in Table 2:

TABLE 2.

j	0	1	2	3	4	5	6	7	8	9	10	11	12
f_j	1	0	0	0	1	1	2	3	7	12	22	40	75
g_j	0	0	0	1	0	1	2	4	6	12	22	41	74

For k even, using the formulae [A'], [B], [C], [D], we have

$$\begin{aligned} \frac{D - A}{4} &= f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2} \\ &\quad + \frac{1}{4} c_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{4} (c_{2k} - 1) (\beta_3 + \gamma_3), \\ \frac{D + A}{4} &= g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2} \\ &\quad + \frac{1}{4} c_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{4} (c_{2k} + 1) (\beta_3 + \gamma_3), \\ \frac{B - C}{4} &= 0, \\ \frac{B + C}{4} &= \frac{1}{2} (\gamma_3 - \beta_3). \end{aligned}$$

Since k is even, $c_{2k+2} = 2f_{2k+2} = 2g_{2k+2}$, $c_{2k} + 1 = 2f_{2k}$, $c_{2k} - 1 = 2g_{2k}$. Hence we obtain the following theorem:

Theorem 2.7. *For k even,*

$$\begin{aligned}
\beta_3\gamma_3 &= \gamma_3\beta_3 = f_0\alpha_0 + f_1\alpha_2 + \cdots f_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{2}f_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{2}(f_{2k} - 1)(\beta_3 + \gamma_3), \\
\beta_3^2 &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{2}g_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{2}g_{2k}\beta_3 + \frac{1}{2}(g_{2k} + 2)\gamma_3, \\
\gamma_3^2 &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} \\
&\quad + \frac{1}{2}g_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{2}(g_{2k} + 2)\beta_3 + \frac{1}{2}g_{2k}\gamma_3.
\end{aligned}$$

All the coefficients of irreducible elements are non-negative integers.

Proof. The only remaining thing to prove is that f_{2k+2} is even, f_{2k} is odd, g_{2j} is even for any j . Since k is even, $c_{2k+2} = 0 \pmod{4}$. Thus $f_{2k+2} = \frac{1}{2}c_{2k+2}$ is even. Likewise $c_{2k} = 1 \pmod{4}$, thus $f_{2k} = \frac{1}{2}(c_{2k} + 1)$ is odd.

$$g_{2j} = \begin{cases} \frac{1}{2}(c_{2j} - 1) & \text{for } j \text{ even,} \\ \frac{1}{2}c_{2j} & \text{for } j \text{ odd} \end{cases}$$

Since $c_{2j} - 1 = 0 \pmod{4}$ for j even, $c_{2j} = 0 \pmod{4}$ for j odd, g_{2j} is even for any j . □

In the same way, we get for k odd:

$$\begin{aligned}
\frac{D-A}{4} &= f_0\alpha_0 + f_1\alpha_2 + \cdots f_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{4}(c_{2k+2} + 1)(\beta_1 + \gamma_1) + \frac{1}{4}c_{2k}(\beta_3 + \gamma_3), \\
\frac{D+A}{4} &= g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{2k+2} \\
&\quad + \frac{1}{4}(c_{2k+2} - 1)(\beta_1 + \gamma_1) + \frac{1}{4}c_{2k}(\beta_3 + \gamma_3), \\
\frac{B-C}{4} &= \frac{1}{2}(\beta_1 - \gamma_1), \\
\frac{B+C}{4} &= 0.
\end{aligned}$$

Since k is odd, $c_{2k+2} + 1 = 2f_{2k+2}$, $c_{2k+2} - 1 = 2g_{2k+2}$, $c_{2k} = 2f_{2k} = 2g_{2k}$. Hence we get:

Theorem 2.8. *For k odd,*

$$\begin{aligned}
\beta_3\gamma_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots f_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{2}(f_{2k+2} + 1)\beta_1 + \frac{1}{2}(f_{2k+2} - 1)\gamma_1 + \frac{1}{2}f_{2k}(\beta_3 + \gamma_3), \\
\gamma_3\beta_3 &= f_0\alpha_0 + f_1\alpha_2 + \cdots f_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{2}(f_{2k+2} - 1)\beta_1 + \frac{1}{2}(f_{2k+2} + 1)\gamma_1 + \frac{1}{2}f_{2k}(\beta_3 + \gamma_3), \\
\beta_3^2 &= \gamma_3^2 = g_0\alpha_0 + g_1\alpha_2 + \cdots + g_{2k+1}\alpha_{4k+2} \\
&\quad + \frac{1}{2}g_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{2}g_{2k}(\beta_3 + \gamma_3)
\end{aligned}$$

All the coefficients of irreducible elements are non-negative integers.

Proof.

It remains to show that f_{2k+2} is odd, f_{2k} is even. In the proof of Theorem 2.7, it has been already proved that g_{2j} is even for any j .

Since k is odd, $c_{2k+2} = 1 \pmod{4}$. Thus $f_{2k+2} - 1 = \frac{1}{2}(c_{2k+2} - 1)$ is even, i.e. f_{2k+2} is odd. Likewise $c_{2k} = 0 \pmod{4}$, thus $f_{2k} = \frac{1}{2}c_{2k}$ is even. \square

Thus far we determined that β_1 and γ_1 are self-conjugate, and computed full irreducible decomposition of β_3, γ_3 , in particular $\overline{\beta_3} = \gamma_3$. This determines the rest of the fusion rule. Note that the conjugate map π on $\mathbb{Z}V_{11}$ is now determined.

First, for $\alpha_{2j}, j = 0, 1, \dots, 2k+1$, the right and left multiplication of α_{2j} on any other object from V_{11} is represented by the matrices $Q_j(\mathbb{D})$ and $Q_j(\pi\mathbb{D}\pi)$ respectively.

Claim 2.9. *The entries of the matrices $R_i(\Delta)$ for $i = 0, 1, \dots, 4k+3$ are non-negative integers. In particular, the entries of the matrices $Q_j(\mathbb{D})$ for $j = 0, 1, \dots, 2k+1$ are non-negative integers.*

Proof.

Immediate from the result in [7], which states that, when Δ is an adjacency matrix of a graph with norm greater than 2, then $R_i(\Delta)$ has non-negative integer entries for any i . \square

It remains to determine the decomposition of tensor product of β_1, γ_1 with themselves and β_3, γ_3 .

Since by the graph $\beta_1 = \beta_3\alpha_2, \gamma_1 = \gamma_3\alpha_2$, the fusion among β_3 and γ_3 together with the fusion of α_2 with all the objects determine $\beta_3\beta_1, \gamma_3\gamma_1, \beta_3\gamma_1, \gamma_3\beta_1$ by imposing associativity. Taking the conjugate, we obtain $\beta_1\beta_3, \gamma_1\gamma_3, \beta_1\gamma_3, \gamma_1\beta_3$ as well. $\beta_1^2 = \beta_1\gamma_3\alpha_2, \gamma_1^2 = \gamma_1\gamma_3\alpha_2, \beta_1\gamma_1 = \beta_1\gamma_3\alpha_2, \gamma_1\beta_1 = \gamma_1\beta_3\alpha_2$ are thus all determined. Since there is no division, subtraction of objects are involved in the process of determining each desired fusion rule, the coefficients are all non-negative integers.

2.2. Fusion rules on ${}_N\mathcal{X}_N \times {}_N\mathcal{X}_M$. We identify ${}_N\mathcal{X}_N$ with V_{11} , ${}_N\mathcal{X}_M$ with V_{12} . From Claim 2.9, $\alpha_i Y$ for i even and any $Y \in V_{12}$ are determined, so are $X\alpha_i$ for $X \in V_{11}$ and i odd. Thus it remains to obtain $\beta_i Y$ and $\gamma_i Y$, where $i = 1, 3, Y = \beta_2$ or γ_2 . They are easily determined, since $\beta_2 = \beta_3\alpha_1, \gamma_2 = \gamma_3\alpha_1$, and the fusion among $\beta_i, \gamma_j, i, j = 1, 3$ are already determined. Here we imposed associativity again. Since the fusion coefficients among β_i 's and γ_j 's are non-negative integers and product of α_1 from the right gives fusion with non-negative integers, the fusion coefficients of $\beta_i Y$ and $\gamma_i Y$ are non-negative integers as well.

2.3. Fusion rules on ${}_N\mathcal{X}_M \times {}_M\mathcal{X}_N$. Let $X \in {}_N\mathcal{X}_M$. Then for j odd,

$$X\bar{\alpha}_j = R_j(\Delta)X.$$

From Claim 2.9, $R_j(\Delta)X$ is a linear combination of the objects in ${}_N\mathcal{X}_N$ with non-negative integer coefficients. It remains to show that $\beta_2\bar{\beta}_2$, $\beta_2\bar{\gamma}_2$, $\gamma_2\bar{\beta}_2$ and $\gamma_2\bar{\gamma}_2$ also have this property. It is immediate, since $\bar{\beta}_2 = \bar{\alpha}_1\bar{\beta}_3$, $\bar{\gamma}_2 = \bar{\alpha}_1\bar{\gamma}_3$, $\beta_2\bar{\alpha} = \beta_1 + \beta_3$, $\gamma_2\bar{\alpha} = \gamma_1 + \gamma_3$, and all the fusion rules involved have decompositions into simple objects with $\mathbb{Z}_{\geq 0}$ -coefficients.

2.4. Fusion rules on ${}_M\mathcal{X}_M \times {}_M\mathcal{X}_M$ and ${}_M\mathcal{X}_M \times {}_M\mathcal{X}_N$. Recall that we have identification ${}_M\mathcal{X}_M = V_{22}$ and ${}_M\mathcal{X}_N = V_{21}$. Let Δ' be the adjacency matrix for Γ' . Then the fusion rules of the tensor products of α'_j 's for $j = 0, 2, \dots, n-1$, as well as $\bar{\alpha}_k$'s for $k = 1, 3, \dots, n-1$ with any objects in $V_{21} \sqcup V_{22}$ are given by the matrices $R_l(\Delta')$, where $l = 0, 1, \dots, n$. Similarly to Claim 2.9, the entries of $R_l(\Delta')$ are all non-negative integers. Furthermore, using Frobenius reciprocity, this also takes care of the coefficients of α'_j 's and $\bar{\alpha}_k$'s in the tensor product of two bimodules.

2.5. Fusion rules on ${}_M\mathcal{X}_M \times {}_M\mathcal{X}_M$. The remaining issue is to determine the fusion rule among f and g . By observing the Perron-Frobenius weights, $\bar{f} = f$, $\bar{g} = g$. Since for j even, all the α'_j 's are self-conjugate as well, we have $fg = gf$.

Theorem 2.10.

$$\begin{aligned} \langle f^2, f \rangle &= d_{2k-1}, \quad \langle fg, f \rangle = d_{2k}, \\ \langle fg, g \rangle &= d_{2k+1}, \quad \langle g^2, g \rangle = d_{2k+2}, \end{aligned}$$

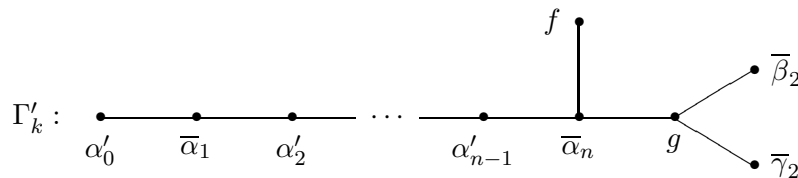
where d_k 's are as in the proof of Claim 2.4, namely defined by $d_j = d_{j-1} + d_{j-2} + d_{j-3}$, $d_{-1} = 0$, $d_0 = d_1 = 1$.

Lemma 2.11.

$$\begin{aligned} \langle f^2, f \rangle - \langle fg, g \rangle &= d_{2k-1} - d_{2k+1}, \\ \langle fg, f \rangle - \langle g^2, g \rangle &= d_{2k} - d_{2k+2}, \\ \langle fg, g \rangle - \langle g^2, g \rangle &= d_{2k+1} - d_{2k+2}. \end{aligned}$$

Proof of Lemma 2.11 We use the similar strategy as in Claim 2.4. Let G' be the adjacency matrix for (V_{22}, V_{21}) corresponding to the graph Γ'_k , and let

$$\Delta' := \begin{pmatrix} 0 & G' \\ G'^t & 0 \end{pmatrix}.$$



Observe

$$\begin{aligned}
R_0(\Delta')(g-f) &= (g-f), \\
R_1(\Delta')(g-f) &= \overline{\gamma}_2 + \overline{\beta}_2, \\
R_2(\Delta')(g-f) &= g+f, \\
R_3(\Delta')(g-f) &= 2\alpha'_n, \\
R_4(\Delta')(g-f) &= 2\alpha'_{n-1} + f + g,
\end{aligned}$$

where $\alpha'_j = \overline{\alpha}_j$ for j odd. Then we have

$$S_j(\Delta')(g-f) = 2\alpha'_{n-j+3}$$

for $j = 3, 4, \dots, n+3$, where the polynomials S_j 's are as defined in the proof of Claim 2.4. On the other hand,

$$\begin{aligned}
g+f &= R_{n+1}(\mathbb{D}')\alpha'_0 \\
&= R_{4k+4}(\mathbb{D}')\alpha'_0 = Q_{2k+2}(\overline{\alpha}_1\alpha_1).
\end{aligned}$$

Using Lemma 2.5,

$$\begin{aligned}
&(g+f)(g-f) \\
&= (d_0S_{2(2k+2)} + d_1S_{2(2k+1)} + \dots + d_{2k}S_4 + d_{2k+1}R_2 + d_{2k-1}R_0)(\Delta')(g-f) \\
&= (\text{linear combination of } \alpha'_* \text{'s}) + d_{2k+1}(g+f) + d_{2k-1}(g-f) \\
&= (\text{linear combination of } \alpha'_* \text{'s}) + (d_{2k+1} + d_{2k-1})g + (d_{2k+1} - d_{2k-1})f.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\langle (g-f)(g+f), g \rangle &= \langle g^2, g \rangle - \langle f^2, g \rangle \\
&= d_{2k+1} + d_{2k-1} = d_{2k+2} - d_{2k}, \\
\langle (g-f)(g+f), f \rangle &= \langle g^2, f \rangle - \langle f^2, f \rangle = d_{2k+1} - d_{2k-1}. \quad (\#1)
\end{aligned}$$

We obtain further information by investigating $R_2(\Delta')(g+f)(g-f)$. Note that $R_2(\Delta')(g+f) = 2\alpha'_{n-1} + f + 3g$. Therefore

$$\begin{aligned}
&R_2(\Delta')(g+f)(g-f) \\
&= (2\alpha'_{n-1} + f + 3g)(g-f) \\
&= 2\alpha'_{n-1}(g-f) + 3g^2 - f^2 - 2fg \\
&= (\alpha'_* \text{'s}) + 2(d_{2k}(g+f) + d_{2k-2}(g-f)) + 3g^2 - f^2 - 2fg \\
&= (\alpha'_* \text{'s}) + 2(d_{2k} + d_{2k-2})g + 2(d_{2k} - d_{2k-2})f + 3g^2 - f^2 - 2fg \quad (\#1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& R_2(\Delta')(g+f)(g-f) \\
&= R_2(\Delta')(2(d_0\alpha'_2 + d_1\alpha'_4 + \cdots + d_{2k}\alpha'_{4k+2})) + (d_{2k+1} + d_{2k-1})R_2(\Delta')g \\
&\quad + (d_{2k+1} - d_{2k-1})R_2(\Delta')f \\
&= (\alpha'_* \text{'s}) + 2d_{2k}(f+g) + (d_{2k+1} + d_{2k-1})(\alpha'_{n-1} + f + 2g) \\
&\quad + (d_{2k+1} - d_{2k-1})(\alpha'_{n-1} + g) \\
&= (\alpha'_* \text{'s}) + (2d_{2k} + d_{2k+1} + d_{2k-1})f + (2d_{2k} + 3d_{2k+1} + d_{2k-1})g. \quad (\#2)
\end{aligned}$$

Comparing (#1) and (#2) we obtain

$$\begin{aligned}
3 < g^2, g > - < f^2, g > - 2 < fg, g > &= 3d_{2k+1} + d_{2k-1} - 2d_{2k-2}, \\
3 < g^2, f > - < f^2, f > - 2 < fg, f > &= d_{2k+1} + d_{2k-1} + 2d_{2k-2} \quad (\text{b2})
\end{aligned}$$

Combining the equations (b1) and (b2) we obtain the statement of the Lemma. Note that we use Frobenius reciprocity such as $< fg, f > = < f^2, g >$ etc. \square

Lemma 2.12.

$$< g^2, g > = d_{2k+2},$$

which implies, together with Lemma 2.11, Theorem 2.10.

Proof Since $g = \bar{\beta}_2\alpha_1 = \bar{\gamma}_2\alpha_1$,

$$2g = (\bar{\beta}_2 + \bar{\gamma}_2)\alpha_1 = \overline{(\beta_3 + \gamma_3)\alpha_1\alpha_1} = \bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1.$$

Also note $\bar{\gamma}_2 = \bar{\gamma}_3\alpha_1 = \bar{\alpha}_1\beta_3$. Therefore

$$\begin{aligned}
4 < g^2, g > &= < \bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1\bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1, \bar{\alpha}_1\beta_3\alpha_1 > \\
&= < \alpha_1\bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1\bar{\alpha}_1(\beta_3 + \gamma_3)\alpha_1\bar{\alpha}_1, \beta_3 > \\
&= < (\beta_3 + \gamma_3)^2(\alpha_1\bar{\alpha}_1)^3, \beta_3 > \\
&= < (\beta_3 + \gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3 >,
\end{aligned}$$

where we used $\alpha_1\bar{\alpha}_1(\beta_3 + \gamma_3) = \beta_1 + \beta_3 + \gamma_1 + \gamma_3 = \overline{\beta_1 + \beta_3 + \gamma_1 + \gamma_3} = \overline{(\beta_3 + \gamma_3)\alpha_1\bar{\alpha}_1} = (\beta_3 + \gamma_3)\alpha_1\bar{\alpha}_1$. By computation using the graph Γ_k , one obtains

$$\beta_3(\alpha_1\bar{\alpha}_1)^3 = 5\beta_3 + 10\beta_1 + 6\alpha_{n-1} + 6\gamma_1 + \alpha_{n-3} + \gamma_3.$$

The formula for $(\beta_3 + \gamma_3)^2$ is given in Claim 2.4. Using it we obtain

$$\begin{aligned}
& < (\beta_3 + \gamma_3)^2, \beta_3(\alpha_1\bar{\alpha}_1)^3 > \\
&= 8c_{2k} + 12c_{2k+1} + 16c_{2k+2} \\
&= 4c_{2k+1} + 8c_{2k+2} + 8c_{2k+3} \\
&= 4c_{2k+2} + 4c_{2k+3} + 4c_{2k+4} = 4c_{2k+5} = 4d_{2k+2}.
\end{aligned}$$

Therefore $< g^2, g > = d_{2k+2}$.

2.6. Fusion rules on ${}_M\mathcal{X}_M \times {}_M\mathcal{X}_N$. The remaining problem is to determine the fusion rule on $\{f, g\} \times \{\bar{\beta}_2, \bar{\gamma}_2\}$.

$$\begin{aligned} \langle f\bar{\beta}_2, \bar{\beta}_2 \rangle &= \langle f, \bar{\beta}_2\beta_2 \rangle = \langle f, \bar{\alpha}_1\beta_3^2\alpha_1 \rangle \\ &= \langle \alpha_1 f\bar{\alpha}_1, \beta_3^2 \rangle = \langle \alpha_n \bar{\alpha}_1, \beta_3^2 \rangle \\ &= \langle \beta_3^2, \beta_1 \rangle + \langle \beta_3^2, \gamma_1 \rangle + \langle \beta_3^2, \alpha_{n-1} \rangle. \end{aligned}$$

Using Theorems 2.7 and 2.8

$$\langle f\bar{\beta}_2, \bar{\beta}_2 \rangle = g_{2k+2} + g_{2k+1}.$$

Both values are non-negative integers. Similarly we obtain

$$\begin{aligned} \langle f\bar{\beta}_2, \bar{\gamma}_2 \rangle &= \langle f\bar{\gamma}_2, \bar{\beta}_2 \rangle = f_{2k+2} + f_{2k+1}, \\ \langle f\bar{\gamma}_2, \bar{\gamma}_2 \rangle &= g_{2k+2} + g_{2k+1}. \end{aligned}$$

$$\begin{aligned} \langle g\bar{\beta}_2, \bar{\beta}_2 \rangle &= \langle \bar{\beta}_2\alpha_1\bar{\beta}_2, \bar{\beta}_2 \rangle = \langle \bar{\alpha}_1\bar{\beta}_3\alpha_1\bar{\alpha}_1\bar{\beta}_3, \bar{\alpha}_1\bar{\beta}_3 \rangle \\ &= \langle \alpha_1\bar{\alpha}_1\gamma_3\alpha_1\bar{\alpha}_1, \gamma_3\beta_3 \rangle = \langle \overline{(\gamma_1 + \gamma_3)\alpha_1\bar{\alpha}_1}, \gamma_3\beta_3 \rangle. \end{aligned}$$

$$\begin{aligned} \overline{(\gamma_1 + \gamma_3)\alpha_1\bar{\alpha}_1} &= (\gamma_1 + \beta_3)\alpha_1\bar{\alpha}_1 \\ &= (\alpha_{n-1} + \beta_1 + 2\gamma_1 + \gamma_3) + \beta_1 + \beta_3 \\ &= \alpha_{n-1} + 2(\beta_1 + \gamma_1) + \gamma_3 + \beta_3 \\ &= \overline{\alpha_{n-1} + 2(\beta_1 + \gamma_1) + \gamma_3 + \beta_3}. \end{aligned}$$

Thus, using Theorems 2.7 and 2.8 we obtain

$$\langle g\bar{\beta}_2, \bar{\beta}_2 \rangle = \begin{cases} f_{2k+1} + 2f_{2k+2} + f_{2k} - 1 & \text{if } k \text{ even} \\ f_{2k+1} + 2f_{2k+2} + f_{2k} & \text{if } k \text{ odd} \end{cases}$$

Similarly,

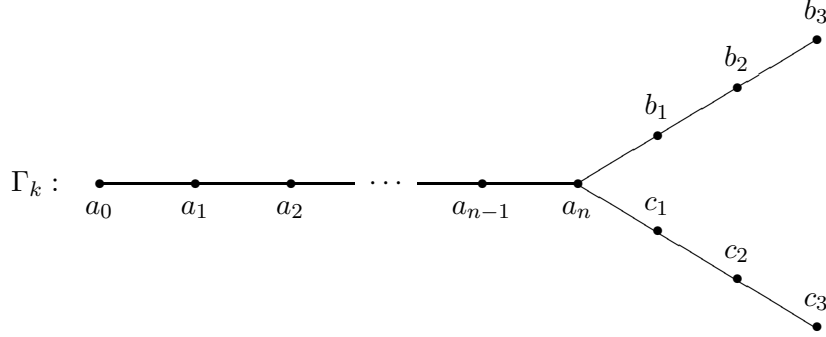
$$\begin{aligned} \langle g\bar{\beta}_2, \bar{\gamma}_2 \rangle &= \langle g\bar{\gamma}_2, \bar{\beta}_2 \rangle \\ &= \begin{cases} g_{2k+1} + 2g_{2k+2} + g_{2k} + 2 & \text{if } k \text{ even} \\ g_{2k+1} + 2g_{2k+2} + g_{2k} & \text{if } k \text{ odd,} \end{cases} \\ \langle g\bar{\gamma}_2, \bar{\gamma}_2 \rangle &= \langle g\bar{\beta}_2, \bar{\beta}_2 \rangle. \end{aligned}$$

3. EXISTENCE OF THE FUSION ALGEBRA

Let $k \in \mathbb{N}_0$, and put $n = 4k + 3$ as before. In this section we will reserve the symbols

$$(\alpha_j)_{0 \leq k \leq n}, (\beta_j)_{1 \leq j \leq 3}, (\gamma_j)_{1 \leq j \leq 3}$$

for elements in a certain bi-graded \mathbb{Z} -algebra \mathcal{A} which we define later. Therefore we relabel the vertices of the graph Γ_k in the following way:



As in Section 2.1, we let G be the adjacency matrix for $(\Gamma_k^{\text{even}}, \Gamma_k^{\text{odd}})$, where

$$\begin{aligned}\Gamma_k^{\text{even}} &= \{a_0, a_2, \dots, a_{n-1}, b_1, c_1, b_3, c_3\}, \\ \Gamma_k^{\text{odd}} &= \{a_1, a_3, \dots, a_n, b_2, c_2\},\end{aligned}$$

we set $\mathbb{D} = GG^t$, and

$$\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}.$$

We set $(q_k)_{k=0}^\infty$ be the sequence of polynomials defined

$$\begin{aligned}q_0(t) &= t^2 - 5t + 3 \\ q_1(t) &= (t-1)(t^3 - 8t^2 + 17t - 5) \\ q_k(t) &= (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2\end{aligned}$$

as in Section 2.1. Then the characteristic polynomial for \mathbb{D} is

$$\chi_k(t) = t^2(t-2)^2q_k(t).$$

(cf. Section 2.1). Moreover $q_k(t)$ is a polynomial of degree $2k+2$ with $2k+2$ distinct roots, because by [3], either $q_k(t)$ or $q_k(t)/(t-1)$ is an irreducible polynomial. From the recursion formula for the q_k -polynomials, one obtains

$$\begin{aligned}q_k(0) &= 2k+3 \\ q_k(2) &= (-1)^{k+1}(2k+3),\end{aligned}$$

In particular, 0 and 2 are not roots of q_k . Let $k \in \mathbb{N}_0$ be now fixed. From the above, we know that $\chi_k(t)$ has exactly $2k+4$ distinct roots $(t_j)_{j=1}^{2k+4}$, where $t_1 = 0$, $t_2 = 2$, and t_3, \dots, t_{2k+4} are the roots of $q_k(t)$. Since $\mathbb{D} = GG^t$ is a positive operator, $t_j \geq 0$ for $1 \leq j \leq 2k+4$.

Lemma 3.1. *Let E_j be the orthogonal projection on the eigenspace of \mathbb{D} corresponding to the eigenvalue t_j ($1 \leq j \leq 2k+4$) and put*

$$\mu_j = \langle E_j a_0, a_0 \rangle,$$

where $\langle \bullet, \bullet \rangle$ is the inner product in $l^2(\Gamma_k^{\text{even}})$. Then

$$(a) \sum_{j=1}^{2k+4} \mu_j = 1,$$

$$(b) \mu_j > 0 \text{ for } 1 \leq j \leq 2k+4,$$

$$(c) \mu_1 = \mu_2 = \frac{1}{2k+3}.$$

Proof.

(a): Since \mathbb{D} is a symmetric matrix, $\sum_{j=1}^{2k+4} E_j = I$, which proves (a).

(b): From Section 2.1, we have

$$\begin{aligned} Q_j(\mathbb{D})a_0 &= R_{2j}(\Delta)a_0 = a_{2j} \quad (0 \leq j \leq 2k+1), \\ Q_{2k+2}(\mathbb{D})a_0 &= R_{4k+4}(\Delta)a_0 = b_1 + c_1, \\ Q_{2k+3}(\mathbb{D})a_0 &= R_{4k+6}(\Delta)a_0 = b_1 + c_1 + b_3 + c_3. \end{aligned}$$

Since $\{a_0, a_2, \dots, a_{4k+2}, b_1 + c_1, b_1 + c_1 + b_3 + c_3\}$ is a set of $2k+4$ linear independent vectors in $l^2(\Gamma_k^{\text{even}})$, and since $(Q_j)_{0 \leq j \leq 2k+3}$ spans the set of polynomials of degree less or equal to $2k+3$, we have

$$P(\mathbb{D})a_0 \neq 0$$

for every non-zero polynomial $P \in \mathbb{R}[x]$ with $\deg(P) \leq 2k+3$. On the other hand, since \mathbb{D} is diagonalisable with eigenvalues $(t_j)_{j=1}^{2k+4}$, we have

$$E_j = P_j(\mathbb{D}),$$

where

$$P_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i}, \quad (t \in \mathbb{R})$$

is a polynomial of degree $2k+3$. Hence

$$\mu_j = \langle E_j a_0, a_0 \rangle = \|E_j a_0\|^2 > 0 \quad (1 \leq j \leq 2k+4).$$

(c): From Section 2.1, we have

$$\begin{aligned} \text{range}(E_1) &= E(\mathbb{D}, 0) = \text{span}\{y_1, y_2\}, \\ \text{range}(E_2) &= E(\mathbb{D}, 2) = \text{span}\{x_1, x_2\}, \end{aligned}$$

where

$$\begin{aligned} x_1 &:= 2(a_0 + a_2) - 2(a_4 + a_6) + \dots + (-1)^k 2(a_{4k} + a_{4k+2}) \\ &\quad + (-1)^{k+1} (b_1 + c_1 + b_3 + c_3) \\ x_2 &:= (b_1 - c_1) + (b_3 - c_3) \\ y_1 &:= 2a_0 - 2a_2 + \dots + 2a_{4k} - 2a_{4k+2} + (b_1 + c_1) - (b_3 + c_3) \\ y_2 &:= (b_1 - c_1) - (b_3 - c_3) \end{aligned}$$

Since $y_1 \perp y_2$ and $y_2 \perp a_0$, we get

$$\mu_1 = \langle E_1 a_0, a_0 \rangle = \frac{|\langle y_1, a_0 \rangle|^2}{\|y_1\|^2} = \frac{1}{2k+3}$$

and similarly

$$\mu_2 = \langle E_2 a_0, a_0 \rangle = \frac{|\langle x_1, a_0 \rangle|^2}{\|x_1\|^2} = \frac{1}{2k+3}.$$

□

Corollary 3.2. *Let $(e_{ij})_{i,j=1}^{2k+4}$ be the matrix units of $M_{2k+4}(\mathbb{R})$. Put*

$$\begin{aligned}\mathcal{B} &= \text{span}_{\mathbb{R}}\{e_{11}, e_{12}, e_{21}, e_{22}, e_{33}, e_{44}, \dots, e_{2k+4, 2k+4}\} \\ &\cong M_2(\mathbb{R}) \oplus l^\infty(\{3, 4, \dots, 2k+4\}, \mathbb{R})\end{aligned}$$

Then \mathcal{B} is a finite dimensional real C^ -algebra and $\mu : \mathcal{B} \rightarrow \mathbb{R}$ given by*

$$\mu(b) := \sum_{j=1}^{2k+4} \mu_j b_{jj}, \quad b = (b_{ij})_{i,j=1}^{2k+4} \in \mathcal{B}$$

is a faithful trace state on \mathcal{B} .

Proof it is clear from (a) and (b) in Lemma 3.1 that μ is a faithful state on \mathcal{B} and the trace property

$$\mu(bc) = \mu(cb), \quad b, c \in \mathcal{B}$$

follows from (c) in Lemma 3.1. □

Lemma 3.3. *Let $k \in \mathbb{N}_0$ be fixed and let $\mu : \mathcal{B} \rightarrow \mathbb{R}$ be the trace defined above, and put*

$$A := \text{diag}(0, \sqrt{2}, \sqrt{t_3}, \dots, \sqrt{t_{2k+4}}),$$

where t_3, \dots, t_{2k+4} are the roots of q_k . Then

(a) *For every even polynomial $P \in \mathbb{R}[x]$*

$$\mu(P(A)) = \langle P(\Delta)a_0, a_0 \rangle.$$

(b) *Let $P, Q \in \mathbb{R}[x]$ be two polynomials, which are either both even or both odd. Then*

$$\mu(P(A)Q(A)) = \langle P(\Delta)a_0, Q(\Delta)a_0 \rangle.$$

(c) *Let $n = 4k + 3$ (as usual), then*

$$R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.$$

Proof

(a): Let $Q \in \mathbb{R}[x]$ be so that $P(t) = Q(t^2)$. Then

$$\langle P(\Delta)a_0, a_0 \rangle = \langle Q(\mathbb{D})a_0, a_0 \rangle.$$

Let E_j denote the spectral projection of \mathbb{D} corresponding to the eigenvalue t_j ($1 \leq j \leq 2k+4$) as before, where $t_1 = 0$ and $t_2 = 2$. Then

$$Q(\mathbb{D}) = \sum_{j=1}^{2k+4} Q(t_j)E_j.$$

Hence

$$\begin{aligned}\langle Q(\mathbb{D})a_0, a_0 \rangle &= \sum_{j=1}^{2k+4} Q(t_j) \langle E_j a_0, a_0 \rangle \\ &= \sum_{j=1}^{2k+4} \mu_j Q(t_j) \\ &= \mu(Q(A^2)) = \mu(P(A)).\end{aligned}$$

(b): Under the assumption on P and Q , the product PQ is an even polynomial. Hence by (a) we have

$$\begin{aligned}\mu(P(A)Q(A)) &= \langle P(\Delta)Q(\Delta)a_0, a_0 \rangle \\ &= \langle P(\Delta)a_0, Q(\Delta)a_0 \rangle.\end{aligned}$$

(c): Put $P = Q = R_{n+4} - R_{n+2} - R_n - R_{n-2}$, which is an odd polynomial. By (b),

$$\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2.$$

From the recursive formula for the polynomials R_j one has

$$\begin{aligned}R_{n-2}(\Delta)a_0 &= a_{n-2}, \\ R_n(\Delta)a_0 &= a_n, \\ R_{n+2}(\Delta)a_0 &= a_n + b_2 + c_2, \\ R_{n+4}(\Delta)a_0 &= a_{n-2} + 2a_n + b_2 + c_2 \\ &= (R_{n+2}(A) + R_n(A) + R_{n-2}(A))a_0.\end{aligned}$$

Hence $\mu(P(A)^2) = \|P(\Delta)a_0\|_2^2 = 0$, and since μ is a faithful trace on \mathcal{B} , we have $P(A) = 0$. \square

Remark 3.4. Since $P = R_{n+4} - R_{n+2} - R_n - R_{n-2}$ is an odd polynomial and $P(A) = 0$, we know that $P(t)$ has at least $n + 4 = 4k + 7$ roots

$$0, \pm\sqrt{2}, \pm\sqrt{t_3}, \dots, \sqrt{t_{2k+4}},$$

which are exactly the distinct roots of $t(t^2 - 2)q_k(t^2)$. Since P and $t(t^2 - 2)q_k(t^2)$ are both monic polynomial of degree $4k + 7$, it follows that

$$(R_{n+4} - R_{n+2} - R_n - R_{n-2})(t) = t(t^2 - 2)q_k(t^2).$$

It is not hard to prove this identity directly by using the recursion formulas for the polynomials $\{q_k\}$'s and $\{R_j\}$'s.

Definition 3.5. Let $k \in \mathbb{N}_0$, $n = 4k + 3$, and let (\mathcal{B}, μ) and $A = \text{diag}(\sqrt{t_1}, \sqrt{t_2}, \dots, \sqrt{t_{2k+4}}) \in \mathcal{B}$ be as before. Let $(f_{ij})_{i,j=1}^2$ be the matrix units in $M_2(\mathbb{R})$, and put

$$V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22},$$

where $V_{ij} \subset \mathcal{B} \otimes f_{ij}$ ($i, j = 1, 2$) are described as below:

a) $V_{11} = \{\alpha_0, \alpha_2, \alpha_4, \dots, \alpha_{4k+2}, \beta_1, \gamma_1, \beta_3, \gamma_3\}$, where

$$\begin{aligned}\alpha_{2j} &= R_{2j}(A) \otimes f_{11}, \quad 0 \leq j \leq 2k + 1, \\ \beta_1 &= \frac{1}{2}(R_{n+1}(A) + \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11}, \\ \gamma_1 &= \frac{1}{2}(R_{n+1}(A) - \sqrt{2k+3}(e_{12} + e_{21})) \otimes f_{11}, \\ \beta_3 &= \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11}, \\ \gamma_3 &= \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) - \sqrt{2k+3}(e_{12} - e_{21})) \otimes f_{11}\end{aligned}$$

b) $V_{12} = \{\alpha_1, \alpha_3, \alpha_5, \dots, \alpha_{4k+3}, \beta_2, \gamma_2\}$ where

$$\begin{aligned}\alpha_{2j+1} &= R_{2j+1}(A) \otimes f_{12}, \quad 0 \leq j \leq 2k+1, \\ \beta_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12}, \\ \gamma_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{12}) \otimes f_{12},\end{aligned}$$

c) $V_{21} = \{\bar{\alpha}_1, \bar{\alpha}_3, \bar{\alpha}_5, \dots, \bar{\alpha}_{4k+3}, \bar{\beta}_2, \bar{\gamma}_2\}$ where

$$\begin{aligned}\bar{\alpha}_{2j+1} &= R_{2j+1}(A) \otimes f_{21}, \quad 0 \leq j \leq 2k+1, \\ \bar{\beta}_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{21}) \otimes f_{21}, \\ \bar{\gamma}_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{21}) \otimes f_{21},\end{aligned}$$

d) $V_{22} = \{\alpha'_0, \alpha'_2, \dots, \alpha'_{4k+2}, f, g\}$ where

$$\begin{aligned}\alpha'_j &= R_{2j}(A) \otimes f_{22}, \quad 0 \leq j \leq 2k+1, \\ f &= \frac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A) \otimes f_{22}, \\ g &= \frac{1}{2}(R_{n+3} - R_{n-1})(A) \otimes f_{22}.\end{aligned}$$

e) The conjugation map $V_{12} \rightarrow V_{21}$ and $V_{21} \rightarrow V_{12}$ is already defined earlier. For V_{11}, V_{22} all the elements are defined to be self-conjugate except β_3 and γ_3 which are defined to be conjugate of each other. Note that for every $X \in V_{ij}$, the conjugate \bar{X} is equal to X^* (or X^t , since all the matrices here are real).

f) We will equip $\mathbb{R}V_{ij} \subset \mathcal{B} \otimes f_{ij}$ with inner products given by

$$\langle b \otimes f_{ij}, c \otimes f_{ij} \rangle_\mu := \mu(c^t b) = \mu(bc^t)$$

for every $b, c \in \mathbb{R}V_{ij}$ ($i, j = 1, 2$).

Lemma 3.6. Let $i, j \in \{1, 2\}$. For $X, Y \in V_{ij}$,

$$\langle X, Y \rangle_\mu = \begin{cases} 1 & \text{if } X = Y, \\ 0 & \text{if } X \neq Y. \end{cases}$$

Proof

Let $(b, c)_\mu := \mu(c^t b) = \mu(bc^t)$, $b, c \in \mathcal{B}$ be the inner product on \mathcal{B} given by μ , and put $\|b\|_\mu(b, b)_\mu^{1/2}$, $b \in \mathcal{B}$.

a): Case $(i, j) = (1, 1)$. It suffices to show that

$$S_1 := \{R_0(A), R_2(A), \dots, R_{n+1}(A), (R_{n+3} - R_{n+1} - R_{n-1})(A), e_{12} + e_{21}, e_{12} - e_{21}\}$$

is an orthogonal set in \mathcal{B} and that

$$\begin{aligned}\|R_{2j}(A)\|_\mu^2 &= 1, \quad 0 \leq j \leq \frac{n-1}{2}, \\ \|R_{n+1}(A)\|_\mu^2 &= 2, \\ \|(R_{n+3} - R_{n+1} - R_{n-1})(A)\|_\mu^2 &= 2, \\ \|e_{12} + e_{21}\|_\mu^2 &= \|e_{12} - e_{21}\|_\mu^2 = \frac{2}{2k+3}.\end{aligned}$$

By the definition of μ in Corollary 3.2, it is clear that $e_{12} + e_{21}$ and $e_{12} - e_{21}$ are μ -orthogonal to the remaining matrices in S_1 , because $R_j(A)$ is a diagonal matrix for all $j \in \mathbb{N}_0$. Moreover, by Lemma 3.1,

$$\begin{aligned} \langle e_{12} + e_{21}, e_{12} - e_{21} \rangle_\mu &= \mu(e_{11} - e_{22}) = \mu_1 - \mu_2 = 0, \\ \|e_{12} + e_{21}\|_\mu^2 &= \|e_{12} - e_{21}\|_\mu^2 = \mu(e_{11} + e_{22}) = \mu_1 + \mu_2 = \frac{2}{2k+3}. \end{aligned}$$

By Lemma 3.3 (b), the remaining part of the proof in the V_{11} -case reduces to show that

$$T_1 := \{R_0(\Delta)a_0, R_2(\Delta)a_0, \dots, R_{n+1}(\Delta)a_0, (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))a_0\}$$

is an orthogonal set in $l^2(\Gamma_k)$ with

$$\begin{aligned} \|R_{2j}(\Delta)a_0\|^2 &= 1, \quad 0 \leq j \leq n-1, \\ \|R_{n+1}(\Delta)a_0\|^2 &= 2, \\ \|(R_{n+3} - R_{n+1} - R_{n-1})(\Delta)a_0\|^2 &= 2. \end{aligned}$$

This follows from the fact that

$$T_1 = \{a_0, a_2, \dots, a_{n-1}, b_1 + c_1, b_3 + c_3\}.$$

b) cases $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. It suffices to show that

$$S_2 := \{R_1(A), R_3(A), \dots, R_n(A), (R_{n+2} - R_n)(A), e_{12}\}$$

is an orthonormal set in \mathcal{B} and that

$$\begin{aligned} \|R_{2j+1}(A)\|_\mu^2 &= 1, \quad 0 \leq j \leq \frac{n-1}{2}, \\ \|(R_{n+2} - R_n)(A)\|_\mu^2 &= 2, \\ \|e_{12}\|_\mu^2 &= \frac{1}{2k+3}. \end{aligned}$$

It is easy to check that e_{12} is orthogonal to the remaining elements of S_2 and that $\|e_{12}\|_\mu^2 = (2k+3)^{-1}$ by Lemma 3.3 (b). The remaining statement about the set S_2 follow from the fact that

$$\begin{aligned} T_2 &= \{R_1(\Delta)a_0, R_3(\Delta)a_0, \dots, R_n(\Delta)a_0, (R_{n+2} - R_n)(\Delta)a_0\} \\ &= \{a_1, a_3, \dots, a_n, b_2 + c_2\} \end{aligned}$$

is an orthonormal set in $l^2(\Gamma_k)$, and that

$$\begin{aligned} \|a_{2j+1}\|^2 &= 1, \quad 0 \leq j \leq \frac{n-1}{2} \\ \|b_2 + c_2\|^2 &= 2. \end{aligned}$$

c) Case $(i, j) = (2, 2)$. The statement follows in this case if we can show that

$$S_3 := \{R_0(A), R_2(A), \dots, R_{n-1}(A), \frac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A), \frac{1}{2}(R_{n+3} - R_{n-1})(A)\}$$

is a μ -orthogonal set in \mathcal{B} . By Lemma 3.3 (b) this reduces to showing that

$$T_3 := \{a_0, a_2, \dots, a_{n-1}, \frac{1}{2}(b_1 + c_1 + b_3 + c_3), \frac{1}{2}(b_1 + c_1 - b_3 - c_3)\}$$

is an orthogonal set in $l^2(\Gamma_k)$, which is obvious. \square

Theorem 3.7. *Let $V = V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$ as in Definition 3.5. Then $\mathbb{Z}V \subset M_2(\mathcal{B})$ form a fusion ring, with coefficients given by*

$$N_{X,Y}^Z = \langle XY, Z \rangle_\mu,$$

where $X \in V_{ij}, Y \in V_{jk}, Z \in V_{ik}, (i, j, k) \in \{1, 2\}^3$, and with units $\alpha_0 \in V_{11}$ and $\alpha'_0 \in V_{22}$. Moreover the graph with vertices $V_{11} \sqcup V_{12}$ obtained by right multiplication by $\alpha = \alpha_1$ is Γ_k and the graph with vertices $V_{21} \sqcup V_{22}$ obtained by right multiplication $\bar{\alpha}$ is Γ'_k .

Proof.

Note that by Lemma 3.6, V_{ij} is a linear independent set in $\mathcal{B} \otimes f_{ij}$ for all $i, j \in \{1, 2\}$. Hence

$$\dim(\mathbb{R}V_{11}) = |V_{11}| = 2k + 6$$

and

$$\dim(\mathbb{R}V_{12}) = \dim(\mathbb{R}V_{21}) = \dim(\mathbb{R}V_{22}) = 2k + 4.$$

This implies that

$$\begin{aligned} \mathbb{R}V_{11} &= \mathcal{B} \otimes f_{11}, \\ \mathbb{R}V_{12} &= \text{span}\{e_{12}, e_{22}, e_{33}, \dots, e_{2k+4, 2k+4}\} \otimes f_{12}, \\ \mathbb{R}V_{21} &= \text{span}\{e_{21}, e_{22}, e_{33}, \dots, e_{2k+4, 2k+4}\} \otimes f_{21}, \\ \mathbb{R}V_{22} &= \text{span}\{e_{11}, e_{22}, e_{33}, \dots, e_{2k+4, 2k+4}\} \otimes f_{22}, \end{aligned}$$

because the four inclusions \subset are obvious, and the right hand sides have dimensions $2k+6$ (resp. $2k+4, 2k+4, 2k+4$). Therefore

$$\mathbb{R}V = \mathbb{R}V_{11} \oplus \mathbb{R}V_{12} \oplus \mathbb{R}V_{21} \oplus \mathbb{R}V_{22}$$

form a bi-graded \mathbb{R} -algebra, and the conjugation $X \rightarrow \bar{X}$ extends by linearity to all of $\mathbb{R}V$ and it is given by transposition of matrices. Moreover, for $X \in V_{ij}, Y \in V_{jk}, (i, j, k \in \{1, 2\})$, we have a unique decomposition

$$XY = \sum_{Z \in V_{ik}} N_{X,Y}^Z Z,$$

where by Lemma 3.6

$$N_{X,Y}^Z = \langle XY, Z \rangle_\mu \in \mathbb{R}.$$

The identities

$$N_{X,Y}^Z = N_{Z,\bar{Y}}^X = N_{\bar{X},Z}^Y = N_{\bar{Z},X}^{\bar{Y}} = N_{Y,\bar{Z}}^{\bar{X}}$$

is now a simple consequence of the fact that μ is a trace state on the real C^* -algebra \mathcal{B} , so in particular

$$\begin{aligned} \mu(b) &= \mu(b^t), \quad b \in \mathcal{B}, \\ \mu(bc) &= \mu(cb), \quad b, c \in \mathcal{B} \end{aligned}$$

It remains to be proved that $N_{X,Y}^Z \in \mathbb{N}_0$ and that multiplication from the right by $\alpha = \alpha_1$ (resp. $\bar{\alpha}$) on V_{11} (resp. V_{22}) generates the graph Γ_k (resp. Γ'_k).

Lemma 3.8. *Let $\alpha = \alpha_1$.*

a) For $X \in V_{11}, Y \in V_{12}$,

$$\langle X\alpha, Y \rangle_\mu = \langle X, Y\bar{\alpha} \rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\alpha, Y \rangle_\mu)_{X \in V_{11}, Y \in V_{12}}$ is the adjacency matrix G_k for Γ_k .

b) For $X \in V_{22}, Y \in V_{21}$,

$$\langle X\bar{\alpha}, Y \rangle_\mu = \langle X, Y\alpha \rangle_\mu \in \mathbb{N}_0,$$

and $(\langle X\bar{\alpha}, Y \rangle_\mu)_{X \in V_{22}, Y \in V_{21}}$ is the adjacency matrix G'_k for Γ'_k .

Proof

This follows from simple computations using Definition 3.5, Lemma 3.6, the recursion formula

$$(\star) \quad tR_n(t) = R_{n+1}(t) + R_{n-1}(t), n \geq 1$$

and the identity from Lemma 3.3(c)

$$(\star\star) \quad R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0 :$$

a) It follows immediately from (\star) that for $1 \leq j \leq 2k+1$,

$$\alpha_{2j}\alpha = \alpha_{2j+1} + \alpha_{2j-1}$$

which shows that $\alpha_{2j} \in V_{11}$ is connected to α_{2j+1} and α_{2j-1} in V_{12} (with simple edges) and not connected to any other $Y \in V_{12}$. To prove that we recover the graph Γ_k this way we just have to check that $\alpha_0\alpha = \alpha_1$, which is obvious, and that $\beta_1\alpha = \alpha_n + \beta_2$, $\beta_3\alpha = \beta_2$. The last one follows from

$$\begin{aligned} \beta_3\alpha &= \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} + e_{21}))A \otimes f_{12} \\ &= \frac{1}{2}(R_{n+4} - 2R_n - R_{n-2})(A) + \sqrt{2(2k+3)}e_{12} \otimes f_{12} \\ &= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12} \\ &= \beta_2, \end{aligned}$$

where we have used (\star) and $(\star\star)$ and the fact that $e_{12}A = \sqrt{2}e_{12}$, $e_{21}A = 0$. The proof of $\beta_1\alpha = \alpha_n + \beta_2$ is similar.

b) To recover the graph Γ_k from $V_{22} \sqcup V_{21}$, it suffices to prove that

$$\begin{aligned} \alpha'_0\bar{\alpha} &= \bar{\alpha}_1, \\ \alpha'_{2j}\bar{\alpha} &= \bar{\alpha}_{2j+1} + \bar{\alpha}_{2j-1} \quad (1 \leq j \leq 2k+1) \\ f\bar{\alpha} &= \bar{\alpha}_n \\ g\bar{\alpha} &= \bar{\alpha}_n + \bar{\beta}_2 + \bar{\gamma}_2 \end{aligned}$$

The first two are obvious. Let us prove $f\bar{\alpha} = \bar{\alpha}_n$. The formula for $g\bar{\alpha}$ is obtained in the same way

$$\begin{aligned} f\bar{\alpha} &= \frac{1}{2}((R_{n-1}(A) + 2R_{n+1}(A) - R_{n+3}(A))A \otimes f_{21} \\ &= \frac{1}{2}(R_{n-2} + 3R_n + R_{n+2} - R_{n+4})(A) \otimes f_{21} \\ &= \frac{1}{2} \cdot 2R_n(A) \otimes f_{21} \\ &= \bar{\alpha}_n \end{aligned}$$

where we again have used (\star) and $(\star\star)$.

Lemma 3.9. *Put*

$$\xi := (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3).$$

Then

$$\bar{\xi} := (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3),$$

and

$$\begin{aligned} \frac{1}{2}\xi\bar{\xi} &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3) \\ \frac{1}{2}\bar{\xi}\xi &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2}) \\ &\quad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3) \end{aligned}$$

Proof.

Clearly $\bar{\xi} = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3)$. By Lemma 3.8, we know that the linear maps

$$R_\alpha : \mathbb{R}V_{11} \rightarrow \mathbb{R}V_{12}$$

$$R_{\bar{\alpha}} : \mathbb{R}V_{12} \rightarrow \mathbb{R}V_{11}$$

obtained by right multiplication by α (resp. $\bar{\alpha}$) have the matrices G^t (resp. G) expressed with respect to bases V_{11} for $\mathbb{R}V_{11}$ and V_{11} for $\mathbb{R}V_{12}$. Hence

$$R_{\alpha\bar{\alpha}} := R_{\bar{\alpha}}R_\alpha : \mathbb{R}V_{11} \rightarrow \mathbb{R}V_{12}$$

has the matrix $\mathbb{D} = GG^t$ with respect to the basis V_{11} for $\mathbb{R}V_{11}$. We can now argue exactly as in Case 1 of Section 2.1 to get

$$\begin{aligned} \xi\bar{\xi} &\in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1, \\ \bar{\xi}\xi &\in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1, \end{aligned}$$

where

$$\begin{aligned} y_1 &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3) \\ x_1 &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k 2(\alpha_{4k} + \alpha_{4k+2}) \\ &\quad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3). \end{aligned}$$

Since $\langle \xi\bar{\xi}, \alpha_0 \rangle_\mu = \langle \bar{\xi}\xi, \alpha_0 \rangle_\mu = \langle \xi, \xi \rangle_\mu = 4$ and $\langle y_1, \alpha_0 \rangle_\mu = \langle x_1, \alpha_0 \rangle_\mu = 2$, it follows that $\xi\bar{\xi} = 2y_1$ and $\bar{\xi}\xi = 2x_1$. \square

End of proof of Theorem 3.7.

It remains to be proved that $N_{X,Y}^Z \in \mathbb{N}_0$ for all $X \in V_{ij}, Y \in V_{jk}$ and $Z \in V_{ik}, (i, j \in \{1, 2, 3\})$. Having established the formulas for $\xi\bar{\xi}$ and $\bar{\xi}\xi$ in Lemma 3.8, the proof of $N_{X,Y}^Z \in \mathbb{N}_0$ can be obtained from Section 2. Using that

$$N_{X,Y}^Z = N_{Z,\bar{Y}}^X = N_{\bar{X},Z}^Y,$$

one gets that if X, Y or Z is one of the elements $(\alpha_j)_{0 \leq j \leq n}, (\alpha'_j)_{0 \leq j \leq n}$ (where $\alpha'_{2k+1} = \bar{\alpha}_{2k+1}$), then $N_{X,Y}^Z$ is an entry of the matrix $R_j(\Delta)$ or $R_j(\Delta')$, which by [7] is a non-negative integer. In the remaining cases, X, Y, Z are compatible and comes from the list

$$\beta_1, \gamma_1, \beta_3, \gamma_3, \beta_2, \gamma_2, \bar{\beta}_2, \bar{\gamma}_2, f, g.$$

For $X, Y, Z \in \{\beta_1, \gamma_1, \beta_3, \gamma_3\}$, we have $N_{X,Y}^Z \in \mathbb{N}_0$ by Theorem 2.7, 2.8, and the remark at the end of Section 2.1. The case $X, Y, Z \in \{f, g\}$ is treated in Theorem 2.10 and the remaining cases can easily be reduced to these two cases by using $\beta_2 = \beta_3\alpha$ and $\gamma_2 = \gamma_3\alpha$ (c.f. Sections 2.2 and 2.6). \square

Remark 3.10. From Definition 3.5, we have

$$\begin{aligned}\xi &= (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3) = 2\sqrt{2k+3}e_{12} \otimes f_{11}, \\ \bar{\xi} &= (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3) = 2\sqrt{2k+3}e_{21} \otimes f_{11}.\end{aligned}$$

Thus

$$\begin{aligned}\xi\bar{\xi} &= 4(2k+3)e_{11} \otimes f_{11} \\ \bar{\xi}\xi &= 4(2k+3)e_{22} \otimes f_{11}.\end{aligned}$$

Since $A = \text{diag}(0, \sqrt{2}, \sqrt{t_3}, \dots, \sqrt{t_{2k+4}})$, where t_3, \dots, t_{2k+4} are the distinct roots of $q_k(t)$, and since $0, 2 \notin \{t_3, \dots, t_{2k+4}\}$, e_{11} and e_{22} are the projections on the eigenspaces for A with eigenvalues 0 and 2 respectively. Using $q_k(0) = 2k+3$ and $q_k(2) = (-1)^{k+1}(2k+3)$, one gets

$$\begin{aligned}(2 - A^2)q_k(A^2) &= 2(2k+3)e_{11} \\ A^2q_k(A^2) &= (-1)^{k+1}(2k+3)e_{22},\end{aligned}$$

because the polynomial $(2-t)q_k(t)$ vanishes at $t=2$ and $t=t_j, 3 \leq j \leq 2k+4$ and has the value $2(2k+3)$ at $t=0$. Similarly $tq_k(t)$ vanishes at $t=0$ and $t=t_j, 3 \leq j \leq 2k+4$ and has the value $(-1)^{k+1}2(2k+3)$ at $t=2$. Hence the following two identities holds:

$$\begin{aligned}\xi\bar{\xi} &= 2(2 - A^2)q_k(A^2) \otimes f_{11} = 2(1_N - \alpha\bar{\alpha})q_k(\alpha\bar{\alpha}) \\ \bar{\xi}\xi &= (-1)^{k+1}2A^2q_k(A^2) \otimes f_{11} = (-1)^{k+1}2\alpha\bar{\alpha}q_k(\alpha\bar{\alpha}),\end{aligned}$$

where $1_N = \alpha_0$ and $\alpha = \alpha_1$. Let Q_j denote as usual the polynomial for which $R_{2j}(t) = Q_j(t^2), t \in \mathbb{R}$. Then by Definition 3.5,

$$\begin{aligned}\alpha_{2j} &= Q_j(\alpha\bar{\alpha}) \\ \beta_1 + \gamma_1 &= Q_{2k+2}(\alpha\bar{\alpha}) \\ \beta_3 + \gamma_3 &= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha\bar{\alpha}).\end{aligned}$$

Hence a more direct proof of Lemma 3.8 can be obtained if the two polynomial identifies (i) and (ii) below holds: Put

$$r_k(t) = (2-t)q_k(t), \quad s_k(t) = (-1)^{k+1}tq_k(t).$$

Then

$$\begin{aligned}(i) \quad r_k &= (2Q_0 - 2Q_1 + \dots + 2Q_{2k} - 2Q_{2k+1}) \\ &\quad + (Q_{2k+1} + 2Q_{2k+2} - Q_{2k+3}) \\ (ii) \quad s_k &= 2(Q_0 + Q_2) - 2(Q_2 + Q_4) + \dots + (-1)^k 2(Q_{2k} + Q_{2k+1}) \\ &\quad + (-1)^{k+1}(Q_{2k+3} - Q_{2k+1}).\end{aligned}$$

These two polynomial identities are actually true, and they can be proved by using the recursion formulas for $(q_k)_{k=0}^\infty$ and $(R_j)_{j=0}^\infty$. \square

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